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# MODULAR SYMMETRIES OF $N = 2$ BLACK HOLES

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## ABSTRACT

We discuss the transformation properties of classical extremal  $N = 2$  black hole solutions in  $S$ - $T$ - $U$  like models under  $S$  and  $T$  duality. Using invariants of (subgroups of) the triality group, which is the symmetry group of the classical BPS mass formula, the transformation properties of the moduli on the event horizon and of the entropy under these transformations become manifest. We also comment on quantum corrections and we make a conjecture for the one-loop corrected entropy.

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## 1 Introduction

During the last years it has become obvious that the structure behind string theories is organized by discrete perturbative and non-perturbative transformations, which are either symmetries of a given string theory or map two different theories into one another. (See [1] for recent reviews and references.)

The pattern of relations that arises this way becomes more and more complicated when the number  $D$  of space-time dimensions and the number  $N$  of supersymmetries is decreased.

The case of  $D = 4, N = 2$  models has proved to be especially interesting because it is rich in structure, but still exactly tractable. The perturbative aspects and the role of symplectic transformations were worked out in [2], [3], [4], [5]. One of the most prominent examples in this class is the so-called  $S$ - $T$ - $U$  model, which has just the minimal number 3 of vector multiplets for a theory coming from a  $D = 6, N = 1$  model by toroidal compactification.<sup>1</sup>

These vector multiplets contain the dilaton/axion  $S$  and the two moduli of the torus,  $T$  and  $U$  as their scalar components.

The  $S$ - $T$ - $U$  model can be obtained from the ten-dimensional heterotic  $E_8 \times E_8$  string by compactification on  $K3 \times T_2$  with instanton numbers (14,10). It is related by well established non-perturbative dualities to the  $I$ IA string compactified on the Calabi-Yau-threefold  $\mathbf{P}_{1,1,2,8,12}(24)$  [6] and to the  $I$ IB string on the mirror, as well as to the type  $I$  superstring by a more recently proposed duality [7]. Moreover, there are self-dualities under perturbative and non-perturbative transformations which can be derived either from the triality of  $D = 4, N = 4$  heterotic,  $I$ IA and  $I$ IB models [8] or from the self-duality of the corresponding heterotic  $D = 6, N = 1$  model [9].

The self-dualities manifest themselves very clearly in the spectrum of BPS states, which is organized by the symplectic structure of local  $N = 2$  supersymmetry. The BPS spectrum consists of both elementary and solitonic states, where the latter ones can be explicitly constructed as extremal black hole solutions of the low energy effective action. (See [10] for a review on stringy extremal black holes.) Extremal  $D = 4, N = 2$  black holes with a non-vanishing event horizon have the property that the moduli take on the horizon certain fixed point values, which solve an extremization problem for the central charge of the  $N = 2$  algebra [11], [12]. (This seems to have generalizations for theories with higher  $D$  and  $N$  [13].) Moreover the absolute value squared of the central charge

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<sup>1</sup>The model has in addition a large number of hypermultiplets which are not relevant for our purpose.

coincides with the extremized ADM mass squared of the extremal black hole and - up to a constant - with its entropy. All these quantities can be expressed in terms of the symplectic quantum numbers of the solution.

The BPS spectrum of the theory receives both perturbative (one-loop) and non-perturbative corrections whose form is restricted by the symplectic structure of local  $N = 2$  supersymmetry. In this note we will focus mostly on duality properties of the classical BPS spectrum, i.e. we consider both elementary and solitonic states but ignore quantum corrections.

As discussed in [8] and in [14], the classical BPS mass formula<sup>2</sup> of the  $S$ - $T$ - $U$  model is invariant under the triality group

$$(SL(2, \mathbf{Z})_S \otimes SL(2, \mathbf{Z})_T \otimes SL(2, \mathbf{Z})_U) \times \mathbf{Z}_2^{T-U} \times \mathbf{Z}_2^{S-T} \times \mathbf{Z}_2^{S-U} , \quad (1.1)$$

where the  $SL(2, \mathbf{Z})$  factors act as fractional linear transformations and the  $\mathbf{Z}_2$  factors act as permutations on the moduli  $S, T, U$ . The spectrum decomposes into certain subsets called orbits, which can be characterized by invariants of the triality group and of certain subgroups as discussed in [14]. The purpose of this note is to use the formalism developed in [14] to make explicit the symmetry and transformation properties of extremal black hole solutions of the  $S$ - $T$ - $U$  model. Note that every  $D = 4, N = 2$  model coming from  $D = 6$  by toroidal compactification will contain the solitons discussed here as a subset. Thus we are discussing the universal sector of all these  $S$ - $T$ - $U$  like models.

The paper is organized as follows: In section 2 the conditions for extremality of the central charge are solved using invariants of subgroups of the triality group. In section 3 the entropy is computed and found to be completely triality invariant. We also comment on the relation to  $D = 4, N = 4$  models. The final sections 4 contains some remarks on quantum corrections. By considering the transformation properties of the classical entropy under one-loop  $T$  duality we arrive at a conjecture for the one-loop corrected entropy. We also speculate on what happens at the non-perturbative level.

## 2 The solution on the horizon

Let us first recall some relevant elements of the symplectic formalism of  $N = 2$  supergravity in the concrete case of supergravity coupled to  $n_V$  vectormultiplets. We follow references [2], [3], [4], [15], which can be consulted for more complete information. The

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<sup>2</sup>Note that this does not imply the existence of all the corresponding states. In fact the full quantum spectrum is not expected to have triality symmetry. See section 4 for a discussion of quantum properties.

low energy effective action of  $N = 2$  supergravity coupled to  $n_V$  vectormultiplets can be written in terms of a so-called symplectic section  $\Omega^T = (P^I, iQ_I)$ ,  $I = 0, \dots, n_V$ . The combined set of field equations and Bianchi identities is invariant under symplectic transformations  $\Gamma \in Sp(2(n_V + 1))$ , which act on the section  $\Omega$  as

$$\begin{pmatrix} P^I \\ iQ_I \end{pmatrix} \rightarrow \Gamma \begin{pmatrix} P^I \\ iQ_I \end{pmatrix} = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \begin{pmatrix} P^I \\ iQ_I \end{pmatrix} . \quad (2.1)$$

Whereas at the classical level the symplectic transformations can be continuous,  $\Gamma \in Sp(2(n_V + 1), \mathbf{R})$ , it is expected that this is broken to a discrete subgroup by instanton effects at the quantum level,  $\Gamma \in Sp(2(n_V + 1), \mathbf{Z})$ .

The mass formula for BPS saturated states is given by

$$M_{BPS}^2 = |z|^2 = |M_I P^I + i N^I Q_I|^2 , \quad (2.2)$$

where  $z$  is the central charge of the  $N = 2$  supersymmetry algebra and  $M_I$  and  $N^I$  are the symplectic quantum numbers. Note that the  $M_I$  are related to electric and the  $N^I$  to magnetic charges under the  $U(1)^{n_V+1}$  gauge group.<sup>3</sup> The BPS mass is invariant under symplectic transformations (2.1) provided the quantum numbers are redefined by  $(N^I, -M_I) \rightarrow (N^I, -M_I)\Gamma^T$ .

The  $2(n_V + 1)$  components of  $\Omega$  can be expressed in terms of the  $n_V$  physical scalar fields, which provide so-called special coordinates on the moduli space. In the case of the  $S$ - $T$ - $U$  model there are three such scalars, namely the dilaton  $S$  and the moduli  $T$  and  $U$ . One choice for  $\Omega$  at the classical level is given by

$$\Omega^T = (P^I, iQ_I) = e^{K/2} (1, TU, iT, iU, iSTU, iS, -SU, -ST) , \quad (2.3)$$

where

$$K = -\log(S + \bar{S})(T + \bar{T})(U + \bar{U}) \quad (2.4)$$

is the Kähler potential. The BPS mass formula then takes the form

$$M_{BPS}^2 = |z|^2 = e^K |\mathcal{M}|^2 , \quad (2.5)$$

where

$$\mathcal{M} = M_0 + M_1 TU + iM_2 T + iM_3 U + iN^0 STU + iN^1 S - N^2 SU - N^3 ST \quad (2.6)$$

is the so-called holomorphic mass.

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<sup>3</sup>The extra  $U(1)$  corresponds to the graviphoton.

Symmetry transformations in  $N = 2$  supergravity coupled to vector multiplets must act as  $Sp(2(n_V + 1), \mathbf{Z})$  transformations. When acting on the section (2.3), symplectic transformations with  $W = Z = 0$  (implying  $V = U^{T,-1}$ ) leave the action invariant, whereas transformations with  $Z = 0$  leave it invariant up to total derivatives. This is the form classical and perturbative symmetries must take. On the other hand transformations with  $Z \neq 0$  are not symmetries of the action and exchange electric and magnetic degrees of freedom, which is the suitable form for non-perturbative symmetries or dualities.

The maximal known symmetry group of the  $S$ - $T$ - $U$  model at the classical level is the triality group [8], [14]

$$(SL(2, \mathbf{Z})_S \otimes SL(2, \mathbf{Z})_T \otimes SL(2, \mathbf{Z})_U) \times \mathbf{Z}_2^{T-U} \times \mathbf{Z}_2^{S-T} \times \mathbf{Z}_2^{S-U} . \quad (2.7)$$

It contains the tree level  $T$  duality group  $O(2, 2, \mathbf{Z})_{T,U} \sim (SL(2, \mathbf{Z})_T \otimes SL(2, \mathbf{Z})_U) \times \mathbf{Z}_2^{T-U}$ , which is a classical symmetry, together with the  $S$  duality group  $SL(2, \mathbf{Z})_S$  [16] and the exchange transformations  $S \leftrightarrow T$  and  $S \leftrightarrow U$ , which are non-perturbative transformations.<sup>4</sup>

As a concrete example let us specify the symplectic matrices realizing the  $S$  duality transformations  $S \rightarrow \frac{aS - ib}{icS + d}$ ,  $T \rightarrow T$ ,  $U \rightarrow U$ :

$$U = d\mathbf{1}, \quad V = a\mathbf{1}, \quad W = b\mathbf{H}, \quad Z = c\mathbf{H} , \quad (2.8)$$

where  $\mathbf{H} = \eta \oplus \eta$  and  $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Also note that the action of the exchange transformations will be given below in (2.12), whereas the matrices of  $O(2, 2, \mathbf{Z})_{T,U}$  transformations can be found in [15].

Let us then consider classical BPS black hole solutions of  $S$ - $T$ - $U$  like  $N = 2$  supergravity coupled to (at least) 3 vector multiplets with four electric and four magnetic charges  $M_0, \dots, N^3$ . In [12] it was argued that the central charge becomes extremal on the horizon. This can be used to express the moduli  $S, T, U$  on the horizon in terms of the quantum number  $M_I, N^I$ . Moreover, the entropy  $\mathcal{S}$  of the black hole is proportional to the absolute value squared of the extremized central charge  $z_{hor}$  and to the extremized ADM mass  $M_{ADM}$  of the black hole:

$$\mathcal{S} = \pi M_{ADM}^2, \quad M_{ADM}^2 = |z_{hor}|^2 = e^K |\mathcal{M}_{hor}|^2 . \quad (2.9)$$

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<sup>4</sup>Since the terminology might be confusing, let us recall that in this context discussing the theory at 'classical' or 'semi-classical' level means that one includes both elementary and solitonic states, but that one ignores all quantum corrections. 'Classical' transformations then leave the action invariant, whereas 'non-perturbative' transformations map it to a dual action which (in the case under consideration) has the same form, but contains a different set of degrees of freedom as the elementary ones.

To be precise, the central charge  $z$  has to be extremized with respect to the Kähler covariant derivative [12], that is by solving<sup>5</sup>

$$D_i z = 0 \rightarrow e^{K/2} (K_i \mathcal{M} + \mathcal{M}_i) = 0, \quad i = S, T, U. \quad (2.10)$$

This yields three quadratic equation for the three moduli  $S, T, U$ :

$$\begin{aligned} \overline{S} &= \frac{M_0 + M_1 T U + i M_2 T + i M_3 U}{i N^0 T U + i N^1 - N^2 U - N^3 T}, \\ \overline{T} &= \frac{M_0 + i M_3 U + i N^1 S - N^2 S U}{M_1 U + i M_2 + i N^0 S U - N^3 S}, \\ \overline{U} &= \frac{M_0 + i M_2 T + i N^1 S - N^3 S T}{M_1 T + i M_3 + i N^0 S T - N^2 S}. \end{aligned} \quad (2.11)$$

The equations are related by the exchange transformations

$$\begin{aligned} \mathbf{Z}_2^{S-T} : \quad & S \leftrightarrow T, \quad M_2 \leftrightarrow N^1, \quad M_1 \leftrightarrow -N^2, \\ \mathbf{Z}_2^{S-U} : \quad & S \leftrightarrow U, \quad M_3 \leftrightarrow N^1, \quad M_1 \leftrightarrow -N^3, \\ \mathbf{Z}_2^{T-U} : \quad & T \leftrightarrow U, \quad M_2 \leftrightarrow M_3, \quad N^2 \leftrightarrow N^3. \end{aligned} \quad (2.12)$$

Thus, it is natural to expect that the extremized moduli as well as the entropy have simple and natural transformation (or invariance) properties under  $S$  and  $T$  duality transformations. As discussed above (the relevant subsector of) the tree level BPS mass formula of  $S$ - $T$ - $U$  like  $N = 2$  models is invariant under the triality group (2.7). In order to make transformation properties manifest, one can use the invariants of certain subgroups of the triality groups, as discussed in appendix A of [14]. For example, taking mutual  $O(2, 2)$  scalar products

$$\langle v, w \rangle = v_0 w_1 + v_1 w_0 + v_2 w_3 + v_3 w_2 \quad (2.13)$$

of the vectors  $M = (M_0, \dots, M_3)$  and  $N = (N^0, \dots, N^3)$  gives rise to the invariants

$$\begin{aligned} \langle M, M \rangle &= 2M_0 M_1 + 2M_2 M_3 \\ \langle N, N \rangle &= 2N^0 N^1 + 2N^2 N^3 \\ \langle M, N \rangle &= M_0 N^1 + M_1 N^0 + M_2 N^3 + M_3 N^2 \end{aligned} \quad (2.14)$$

of the  $T$  duality group

$$O(2, 2, \mathbf{Z})_{(T, U)} = (SL(2, \mathbf{Z})_T \otimes SL(2, \mathbf{Z})_U) \times \mathbf{Z}_2^{T-U}. \quad (2.15)$$

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<sup>5</sup> The lower indices indicate ordinary partial derivatives with respect to the special coordinates  $S, T, U$ .

The exchange symmetries  $S \leftrightarrow T$  and  $S \leftrightarrow U$  map the vectors  $M, N$  to vectors  $M', N'$  and  $M'', N''$  whose components can be read off from (2.12). Using these vectors one obtains the invariants  $\langle M', M' \rangle$ , etc. and  $\langle M'', M'' \rangle$ , etc. of the groups  $O(2, 2, \mathbf{Z})_{S,U} \sim (SL(2, \mathbf{Z})_S \otimes SL(2, \mathbf{Z})_U) \times \mathbf{Z}_2^{S-U}$  and  $O(2, 2, \mathbf{Z})_{S,T} \sim (SL(2, \mathbf{Z})_S \otimes SL(2, \mathbf{Z})_T) \times \mathbf{Z}_2^{S-U}$ . These groups are the classical symmetries of the triality rotated actions, where  $T$  or  $U$  have taken over the role of the dilaton.

Moreover, there is a  $GL(4)$  subgroup of  $Sp(8)$  defined by  $W = Z = 0$  in (2.1). In this subgroup  $V = U^{T,-1}$  and therefore

$$M \cdot N = M_0 N^0 + \dots + M_3 N^3 \quad (2.16)$$

is invariant. Since the tree level  $T$  duality group  $O(2, 2, \mathbf{Z})_{T,U}$  is a subgroup of this  $GL(4)$  as already mentioned above (see [15] for an explicit embedding),  $M \cdot N$  is therefore an  $O(2, 2, \mathbf{Z})_{T,U}$  invariant. Likewise  $M' \cdot N'$  and  $M'' \cdot N''$  are invariant under  $O(2, 2, \mathbf{Z})_{S,U}$  and  $O(2, 2, \mathbf{Z})_{S,T}$ , respectively.

When solving the three quadratic equations (2.11) one encounters the quantity

$$D = \langle M, M \rangle \langle N, N \rangle - (M \cdot N)^2 \quad (2.17)$$

and its transformed  $D'$  and  $D''$  under  $S \leftrightarrow T$  and  $S \leftrightarrow U$  as discriminants.  $D$  is manifestly invariant under  $O(2, 2, \mathbf{Z})_{T,U}$  but one can easily check that it is also invariant under the exchange transformations  $S \leftrightarrow T$  and  $S \leftrightarrow U$  and therefore under the full triality group:  $D = D' = D''$ . Note that it was argued in [14] that no quadratic invariant of the full triality group can be constructed out of  $M_I, N^I$ . This is no contradiction to the result here because the invariant  $D$  is quartic.

In order to obtain extremized moduli with a positive real part, one has to demand that  $D > 0$ . For  $D \leq 0$  the moduli become purely imaginary. This does only make sense if they are imaginary rational,  $S, T, U \in i\mathbf{Q}$ . These values are cusps of the corresponding  $SL(2, \mathbf{Z})$  and are thus equivalent to  $S = T = U = \infty$  which, of course, is not an interesting solution. For  $D > 0$  the explicit solution of (2.11) is

$$\begin{aligned} S &= i \frac{M \cdot N}{\langle N, N \rangle} + \sqrt{\frac{\langle M, M \rangle}{\langle N, N \rangle} - \frac{(M \cdot N)^2}{\langle N, N \rangle^2}} \ , \\ T &= i \frac{M' \cdot N'}{\langle N', N' \rangle} + \sqrt{\frac{\langle M', M' \rangle}{\langle N', N' \rangle} - \frac{(M' \cdot N')^2}{\langle N', N' \rangle^2}} \ , \\ U &= i \frac{M'' \cdot N''}{\langle N'', N'' \rangle} + \sqrt{\frac{\langle M'', M'' \rangle}{\langle N'', N'' \rangle} - \frac{(M'' \cdot N'')^2}{\langle N'', N'' \rangle^2}} \ . \end{aligned} \quad (2.18)$$

The transformation properties of the solutions under triality transformations are manifest: the solution for  $S$  is invariant under the  $T$  duality group  $O(2, 2, \mathbf{Z})_{T,U}$ , whereas  $T$  and  $U$  are exchanged under  $T \leftrightarrow U$  (which acts as  $M' \leftrightarrow M''$ ,  $N' \leftrightarrow N''$  on the quantum numbers) and transform fractional linearly under  $SL(2, \mathbf{Z})_T$  and  $SL(2, \mathbf{Z})_U$ . On the other hand the solutions for  $S$  and  $T$  and  $S$  and  $U$  are exchanged by  $\mathbf{Z}_2^{S-T}$  and  $\mathbf{Z}_2^{S-U}$  respectively. And the solutions for  $T$  and  $U$  are invariant under the groups  $O(2, 2, \mathbf{Z})_{S,U}$ ,  $O(2, 2, \mathbf{Z})_{S,T}$ .

### 3 The entropy

It is now straightforward to compute the entropy with the result

$$\mathcal{S}/\pi = M_{ADM}^2 = \sqrt{\langle M, M \rangle \langle N, N \rangle - (M \cdot N)^2} = \langle N, N \rangle \Re S \quad . \quad (3.1)$$

This agrees with the result obtained in [17] by considering so called double extreme solutions, in which the moduli take the same value at infinity and on the horizon (and are therefore constant in between). As shown above the classical entropy is invariant under the full triality. This had to be expected because the classical BPS spectrum from which it is computed has this property.

From (3.1) one can easily read off that the entropy vanishes for certain classes of black holes.

The first class consists of black holes with only four non-vanishing quantum numbers, including the cases of purely electric ( $N = 0$ ) and purely magnetic ( $M = 0$ ) black holes. In fact, it is a well known property of extremal  $N = 2$  black holes that only dyonic ones can have a non-vanishing event horizon and entropy [10]. By triality we know that the entropy must also vanish if  $M' = 0$  or  $N' = 0$  or  $M'' = 0$  or  $N'' = 0$ .

The second class of black holes with vanishing entropy has eight non-vanishing quantum number which are, however, not completely independent from one another, with the effect that the quantum numbers can be expressed in terms of momentum quantum numbers  $m_i$  ( $i = 1, 2$ ) and winding quantum numbers  $n^i$  of strings around the two-torus as

$$M_I/p = (m_2, -n_2, n_1, -m_1), \quad N^I/q = (-n_2, m_2, -m_1, n_1) \quad . \quad (3.2)$$

This implies that

$$\begin{aligned} \langle M, M \rangle &\equiv -2p^2 n^T m \quad , \\ \langle N, N \rangle &\equiv -2q^2 n^T m \quad , \\ M \cdot N &\equiv -2pq n^T m \quad , \end{aligned} \quad (3.3)$$



and therefore  $\mathcal{S} = 0$ . The corresponding BPS states are those which can be embedded into a heterotic  $D = 4, N = 4$  model as short multiplets [14], implying that only those states which are intermediate from the  $N = 4$  point of view contribute to the entropy. By triality the entropy also vanishes for those black holes, where the vectors  $M', N'$  and  $M'', N''$  take the form (3.3). In  $D = 4, N = 4$  theories the exchange symmetries  $S \leftrightarrow T$  and  $S \leftrightarrow U$  do not act as a self-duality but map heterotic to *IIA* and *IIB* strings, respectively [8], [14]. In particular they map short multiplets of the heterotic theory to multiplets which are intermediate in the heterotic but short in the *IIA* or *IIB* theory [18], [14]. Therefore, the triality rotated  $N = 2$  black holes with vanishing entropies are those which come from short multiplets of the  $N = 4$  *IIA* or *IIB* theory.

Finally we would like to recall the entropy formula for heterotic  $N = 4$  black holes [19]

$$\mathcal{S}/\pi = \sqrt{P^2 Q^2 - (Q \cdot P)^2} \ , \quad (3.4)$$

where the  $Q$  denote 28 electric charges and where the  $P$  denote 28 magnetic charges which lay in the  $(6, 22)$  Narain lattice.  $O(6, 22, \mathbf{Z})$  invariance then restricts the entropy to be of the form (3.4). Comparing to (3.1), it is evident that the  $N = 2$  formula should result from a suitable truncation. Short  $N = 4$  multiplets are characterized by  $P$  and  $Q$  being parallel. Thus it is evident that only intermediate  $N = 4$  multiplets contribute to the entropy.

## 4 Quantum Corrections

In  $N = 2$  supergravity coupled to vector multiplets, there are both perturbative and non-perturbative quantum corrections. The full prepotential of the  $S$ - $T$ - $U$  model takes the form [2], [3], [4]

$$F(S, T, U) = -STU + f(T, U) + f^{(NP)}(e^{-2\pi S}, T, U). \quad (4.1)$$

Note that the perturbative correction  $f(T, U)$  does not depend on the dilaton, reflecting the fact that perturbative corrections can occur at the one-loop level, only.

Let us first neglect non-perturbative corrections and consider the one-loop effects. Obviously, the full perturbative theory with prepotential  $F^{pert}(S, T, U) = -STU + f(T, U)$  is not triality symmetric. Moreover  $T$ -duality is still a symmetry, but the symplectic transformations get modified such that the perturbative  $T$  duality group is no longer  $O(2, 2, \mathbf{Z})_{T,U}$ .

In order to obtain a convenient description of the perturbative  $T$  duality group, a transformation from the symplectic section  $(X^I, i\partial_I F)$  (where  $(X^I) = e^{K/2}(1, iS, iT, iU)$ ),

defined in terms of the prepotential, to the section  $(P^I, iQ_I)$  introduced earlier by means of the symplectic transformation  $P^1 = -iF_1(=TU)$ ,  $iQ_1 = X^1(=iS)$  is required [2], [3].

In the new parametrization, the one-loop  $T$  duality transformations take the form

$$\Gamma^{Tree} = \begin{pmatrix} U & 0 \\ 0 & U^{T,-1} \end{pmatrix} \longrightarrow \Gamma^{Pert} = \begin{pmatrix} U & 0 \\ W & U^{T,-1} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^{T,-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Lambda & 1 \end{pmatrix}, \quad (4.2)$$

where  $\Lambda$  is a symmetric integral matrix which encodes the quantum corrections.

The one-loop transformation rule of symplectic quantum numbers implied by the general formula  $(N, -M) \rightarrow (N, -M)\Gamma^T$  is

$$M \rightarrow U^{T,-1}M - WN, \quad N \rightarrow UN. \quad (4.3)$$

Explicit generators of the perturbative  $T$  duality group have been given in [3], [4], [15]. The most obvious effect is that the dilaton  $S$  is no longer invariant under perturbative  $T$  duality but is shifted by a function of  $T$  and  $U$ . One can however define an invariant dilaton  $S_{invar}$  by adding a suitable function of  $T$  and  $U$  to the dilaton  $S$  [3]. Note that the invariant dilaton is not a special  $N = 2$  coordinate.

The transformation law of the classical entropy under one-loop  $T$ -duality transformations can be easily worked out and reads

$$\mathcal{S} = \pi\sqrt{D} \rightarrow \tilde{\mathcal{S}} = \pi\sqrt{\tilde{D}}, \quad (4.4)$$

where

$$\begin{aligned} \tilde{D} - D &= (\langle WN, WN \rangle - 2\langle WN, U^{T,-1}M \rangle)\langle N, N \rangle \\ &+ 2(M \cdot N)(WN \cdot UN) - (WN \cdot UN)^2. \end{aligned} \quad (4.5)$$

Thus, the tree level expression for the entropy is not invariant under one-loop  $T$  duality. Whereas the breaking of the full triality symmetry by loop corrections is no surprise, one expects  $T$  duality to be true at the perturbative level and thus there must exist a modification of the entropy formula. One can also check that the left and right hand sides of the classical solutions (2.18) for the moduli on the horizon transform differently under perturbative  $T$  duality. This implies that the solutions must be modified at the one-loop level, too. We expect that invariants of the perturbative  $T$  duality group will play a crucial role. Note that  $\langle N, N \rangle$  is such an invariant whereas  $\langle M, M \rangle$ ,  $M \cdot N$  and  $D$  are not.

A natural candidate for the one-loop entropy is found by observing that the tree level entropy can be written as

$$\mathcal{S}_{tree} = \pi \langle N, N \rangle \Re S \quad . \quad (4.6)$$

Now,  $\langle N, N \rangle$  is invariant under one-loop  $T$  duality, and it is well known that one can make the dilaton invariant by adding a suitable function of  $T$  and  $U$ , yielding the so-called invariant dilaton  $S_{invar}$  [3]. This motivates us to conjecture that the one-loop entropy is given by

$$\mathcal{S}_{1-loop} = \pi \langle N, N \rangle \Re S_{invar} \quad . \quad (4.7)$$

The conjecture is also compatible with the known perturbative structure of gravitational threshold corrections. Since at the tree level only states, which are intermediate from the  $N = 4$  point of view contribute to the entropy we expect that at one-loop all contributions of such states go into the invariant dilaton.

Let us finally recall what is known about non-perturbative effects. It is firmly established that the heterotic  $S$ - $T$ - $U$  model is dual to the  $I$ IA compactification on the Calabi-Yau threefold  $\mathbf{P}_{1,1,2,8,12}(24)$  [6], [20], [21], [22], [23], [24]. This implies that the full non-perturbative prepotential is given by the classical prepotential of the dual  $I$ IA model. Symplectic matrices corresponding to the true quantum symmetries can be computed by studying the monodromy properties of the prepotential around its singular loci on the moduli space of the threefold.

Both the structure of the conifold locus of the threefold and an analysis of the perturbative monodromies of the heterotic theory indicate the following:  $T$  duality transformations corresponding to Weyl transformations of generically Higgsed non-Abelian gauge groups (like for instance the  $T \leftrightarrow U$  exchange, which is the Coxeter twist of the  $SU(2)$  gauge group unbroken at  $T = U$ ) are replaced by two non-perturbative quantum monodromies caused by dyons that become massless. This is the stringy generalization of the Seiberg-Witten solution of the pure  $N = 2$   $SU(2)$  super Yang-Mills theory.

Since the Calabi-Yau moduli space contains more singular loci, there is space for other non-perturbative effects as well. There exists, for example, the so-called strong coupling locus [25], which is fixed under the exchange transformation  $S \leftrightarrow T$  [20], [14], [24]. Thus, the  $S \leftrightarrow T$  exchange is, although not a symmetry at the perturbative level, a symmetry of the full non-perturbative theory.

Since the group structure of symmetries is strongly modified when going from the perturbative to the non-perturbative level, one expects that the entropy formula is also further modified. A better knowledge of the monodromy group and of its invariants should be useful for dealing with this question.

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